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Mass Density of *Dp*-branes

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It is shown that the generally covariant definition of mass equals the ADM mass for *Dp*-branes.

As dynamical objects, *Dp*-branes play an important role in the study of dualities in string theories [1,2]. The bosonic sector of the actions in the effective low-energy theory for 10-dimensional type II strings and 11-dimensional supergravity is of the general form

$$I = \int d^D x \sqrt{-g} \left[R - \frac{1}{2} \nabla_M \phi \nabla^M \phi - \frac{1}{2n!} e^{a\phi} F^2_{[n]} \right] \tag{1}$$

The p-brane solutions are those for (1) which possess $(\text{Poincaré})_d \otimes SO(D - d)$ symmetry, $d = p + 1$. Let the spacetime coordinates be split into two ranges: $x^M = (x^\mu, y^m)$, where x^μ ($\mu = 0, 1, \ldots, p$) are coordinates adapted to the $(\text{Poincaré})_d$ isometries on the worldvolume and where y^m ($m = d, \ldots, D - 1$) are the coordinates *transverse* to the worldvolume. The Ansatz for the spacetime metric is

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dy^m dy^n \delta_{mn} \tag{2}$$

where $r = \sqrt{y^m y^m}$ is the isotropic radial coordinate in the transverse space. The desired symmetry $(\text{Poincaré})_d \otimes SO(D - d)$ is guaranteed, as is obvious. A notable feature of the classical p-branes with appropriate electric or mag-

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netic Ansatz is that the are BPS saturated, in the sense that their ADM mass equals the charge involved. This is very similar to the usual extreme Reissner–Nordstrom solution, which can be called a 0-brane [3], in general relativity. Since the the BPS saturated states are characterized by the fact that they preserve some of the full supersymmetries and the energy-momentum P_a involved in the supersymmetric algebra should have a coordinate-independent definition, while the ADM mass definition is not covariant, it is therefore nontrivial to check the BPS saturation by a covariant definition of energy-momentum. We have such a definition at hand.

Using the general translation $\delta x^\mu = e_a^\mu b^a$ in curved spacetime and the Nöther theorem, a generally covariant definition of energy-momentum in the vierbein formalism of Einstein general relativity was obtained in ref. 4. The outline is as follows. Suppose that the spacetime is of dimension D and the Lagrangian is in the first-order formalism [μ denotes the Riemann indices and a the tangent indices in Eqs. (3)–(23)]

$$I = \int_G \mathscr{L}(\phi^A, \partial_\mu \phi^A)\, d^D x \tag{3}$$

where ϕ^A denotes the generic fields. If the action is invariant under the infinitesimal transforms

$$x'^\mu = x^\mu + \delta x^\mu, \qquad \phi'^A(x') = \phi^A(x) + \delta\phi^A(x) \tag{4}$$

(it is not required that $\delta\phi^A_{\partial G} = 0$; see ref. 5), then the following relation holds:

$$\partial_\mu\left(\mathscr{L}\delta x^\mu + \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^A}\, \delta_0 \phi^A\right) + [\mathscr{L}]_{\phi^A}\delta_0\phi^A = 0 \tag{5}$$

where

$$[\mathscr{L}]_{\phi^A} = \frac{\partial \mathscr{L}}{\partial \phi^A} - \partial_\mu \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^A} \tag{6}$$

and $\delta_0\phi^A$ is the Lie variation of ϕ^A,

$$\delta_0\phi^A = \phi'^A(x) - \phi^A(x) = \delta\phi^A(x) - \partial_\mu\phi^A\delta x^\mu \tag{7}$$

If $\mathscr{L}$ is the total Lagrangian of the system, the field equations of ϕ^A is just $[\mathscr{L}]_{\phi^A} = 0$. Hence from Eq. (5) we can obtain the conservation equation corresponding to transforms (4),

$$\partial_\mu \left(\mathscr{L}\delta x^\mu + \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^A}\, \delta_0 \phi^A\right) = 0 \tag{8}$$

It is important to recognize that if $\mathscr{L}$ is not the total Lagrangian, e.g., the gravitational part $\mathscr{L}_g$, then so long as the action of $\mathscr{L}_g$ remains invariant under transforms (4), Eq. (5) is still valid, yet Eq. (8) is no longer admissible because of $[\mathscr{L}_g]_{\phi^A} \neq 0$.

Suppose that ϕ^A denotes the Riemann tensors ϕ^A_μ and Riemann scalars ψ^A; Eq. (5) reads

$$\partial_\mu \left(\mathscr{L}_g \delta x^\mu + \frac{\partial \mathscr{L}_g}{\partial \partial_\mu \phi^A_\nu} \delta_0 \phi^A_\nu \right) + [\mathscr{L}_g]_{\phi^A_\mu} \delta_0 \phi^A_\mu = 0 \tag{9}$$

Under transforms (4), the Lie variations are

$$\delta_0 \phi^A_\nu = -\delta x^\alpha_{,\nu} \phi^A_\alpha - \phi^A_{\nu,\alpha} \delta x^\alpha \tag{10}$$

where the dot comma denotes partial derivative. So Eq. (9) reads

$$\partial_\mu \left[\mathscr{L}_g \delta x^\mu - \frac{\partial \mathscr{L}_g}{\partial \partial_\mu \phi^A_\lambda} (\delta x^\nu_{,\lambda} \phi^A_\nu + \phi^A_{\lambda,\nu} \delta x^\nu) \right] - [\mathscr{L}_g]_{\phi^A_\lambda} (\delta x^\nu_{,\lambda} \phi^A_\nu + \phi^A_{\lambda,\nu} \delta x^\nu) = 0 \tag{11}$$

Comparing the coefficients of δx^ν, $\delta x^\nu_{,\lambda}$, and $\delta x^\nu_{,\mu\lambda}$, we obtain

$$\partial_\lambda ([\mathscr{L}_g]_{\phi^A_\lambda} \phi^A_\nu) = [\mathscr{L}_g]_{\phi^A_\lambda} \phi^A_{\lambda,\nu} \tag{12}$$

Then Eq. (11) can be written as

$$\partial_\mu \left[\mathscr{L}_g \delta x^\mu - \frac{\partial \mathscr{L}_g}{\partial \partial_\mu \phi^A_\lambda} (\delta x^\nu_{,\lambda} \phi^A_\nu + \phi^A_{\lambda,\nu} \delta x^\nu) - [\mathscr{L}_g]_{\phi^A_\mu} \phi^A_\nu \delta x^\nu \right] = 0 \tag{13}$$

or

$$\partial_\mu \left[\left(\mathscr{L}_g \delta^\mu_\nu - \frac{\partial \mathscr{L}_g}{\partial \partial_\mu \phi^A_\lambda} \phi^A_{\lambda,\nu} - [\mathscr{L}_g]_{\phi^A_\mu} \phi^A_\nu \right) \delta x^\nu - \frac{\partial \mathscr{L}_g}{\partial \phi^A_{\lambda,\nu}} \phi^A_\nu \delta x^\nu_{,\lambda} \right] = 0 \tag{14}$$

By definition, we introduce

$$\tilde{I}^\mu_\nu = -\left(\mathscr{L}_g \delta^\mu_\nu - \frac{\partial \mathscr{L}_g}{\partial \partial_\mu \phi^A_\lambda} \phi^A_{\lambda,\nu} - [\mathscr{L}_g]_{\phi^A_\mu} \phi^A_\nu \right) \tag{15}$$

$$\tilde{V}^{\lambda\mu}_\nu = \frac{\partial \mathscr{L}_g}{\partial \phi^A_{\lambda,\mu}} \phi^A_\nu \tag{16}$$

Then Eq. (14) gives

$$\partial_\mu (\tilde{I}^\mu_\nu \delta x^\nu + \tilde{V}^{\lambda\mu}_\nu \delta x^\nu_{,\lambda}) = 0 \tag{17}$$

So, by comparing the coefficients of δx^ν, $\delta x^\nu_{,\mu}$, and $\delta x^\nu_{,\mu\lambda}$, we have the following from Eq. (15):

$$\partial_\mu \tilde{I}^\mu_\nu = 0 \tag{18}$$

$$\tilde{I}^\lambda_\nu = -\partial_\mu \tilde{V}^{\lambda\mu}_\nu, \qquad \tilde{V}^{\mu\lambda}_\nu = -\tilde{V}^{\lambda\mu}_\nu \tag{19}$$

Now suppose that $\delta x^\mu = \epsilon\xi^\mu(x)$, with ϵ an infinitesiaml constant parameter, and $\xi^\mu(x)$ an arbitrary vector. Then it follows from Eqs. (17)–(19) that

$$\partial_\mu \tilde{j}^\mu(\xi) = 0 \tag{20}$$

where

$$\tilde{j}^\mu(\xi) = \partial_\nu \tilde{V}^{\nu\mu} \tag{21}$$

and

$$\tilde{V}^{\nu\mu} = \tilde{V}^{\nu\mu}_\alpha \xi^\alpha \tag{22}$$

Accordingly, we have the conserved charge associated with ξ,

$$Q[\xi] = \int_\Sigma \tilde{j}^0 \, d^2x = \int_{\partial\Sigma} \tilde{V}^{i0} \epsilon_{ij} \, dx^j \tag{23}$$

If we choose $\xi^\mu = e^\mu_a \epsilon^a$, ϵ^a = const, we can obtain the energy-momentum.This definition has a number of advantages over noncovariant ones such as the definitions of Landau and Einstein, etc. [6]. Applying this definition to the $N = 1$ self-dual supergravity, the correct superalgebra is restored [7].

For the action (1), the conservation law reads (we adopt the same convention for indices as in ref. 1)

$$T^M_{\underline{M}} + t^M_{\underline{M}} = \nabla_N V^{MN}_{\underline{M}}, \qquad \partial_M[\sqrt{-g}(T^M_{\underline{M}} + t^M_{\underline{M}})] = 0 \tag{24}$$

$$V^{MN}_{\underline{M}} = 2[e^M_{\underline{R}} e^N_{\underline{S}} \omega_{\underline{M}}{}^{\underline{RS}} + (e^M_{\underline{M}} e^N_{\underline{R}} - e^M_{\underline{R}} e^N_{\underline{M}})\omega^{\underline{R}}] \tag{25}$$

where $T^M_{\underline{M}} + t^M_{\underline{M}}$ is the total energy-momentum tensor and the conservative total energy (mass) is

$$\begin{aligned} E &= \int (T^0_{\underline{0}} + t^0_{\underline{0}})\sqrt{-g} \, d^p\mathbf{x} \, d^{D-d}\mathbf{y} \\ &= \int \nabla_N V^{0N}_{\underline{0}} \sqrt{-g} \, d^p\mathbf{x} \, d^{D-d}\mathbf{y} \\ &= \int \partial_N \tilde{\mathbf{V}}^{0N}_{\underline{0}} \, d^p\mathbf{x} \, d^{D-d}\mathbf{y} \end{aligned} \tag{26}$$

where $\hat{V}^{0N}_{\underline{0}} = V^{0N}_{\underline{0}}\sqrt{-g}$, $T^M_{\underline{M}}$ and $t^M_{\underline{M}}$ are the energy-momentum current for matter and gravity, respectively, and $\omega^{\underline{M}} = e^{\underline{RS}}\omega_{\underline{RMS}}$. For our purpose, the vierbein 1-forms for the metric ansatz are [1]

$$e^{\underline{\mu}} = e^{A(r)}\, dx^{\mu}, \qquad e^{\underline{m}} = e^{B(r)}\, dy^{m} \tag{27}$$

and the spin-connection 1-forms are

$$\omega^{\underline{\mu\nu}} = 0, \qquad \omega^{\underline{\mu n}} = e^{-B(r)}\partial_n A(r) e^{\underline{\mu}} \tag{28}$$

$$\omega^{\underline{mn}} = e^{-B(r)}\partial_n B(r) e^{\underline{m}} - e^{-B(r)}\partial_m B(r) e^{\underline{n}}$$

Thus we have by direct calculation

$$\omega^{\underline{\mu}} = 0, \qquad \omega^{\underline{m}} = -de^{-B(r)}\partial_m A(r) + e^{-B(r)}(1 + d - D)\partial_m B(r) \tag{29}$$

Therefore

$$V^{0m}_{\underline{0}} = 2e^{-(A+2B)}[(1 - d)\partial_m A - (1 + \tilde{d})\partial_m B] \tag{30}$$

where $\tilde{d} = D - d - 2$. For the specific solution

$$A(r) = -\frac{2\tilde{d}}{\Delta(D - 2)} \ln H, \qquad B(r) = -\frac{2}{\Delta(D - 2)} \ln H, \qquad H = 1 + \frac{k}{r^{\tilde{d}}} \tag{31}$$

we have

$$V^{0m}_{\underline{0}} = \frac{4k\tilde{d}}{\Delta}\, e^{-(A+2B)}\, \frac{1}{H}\, \frac{y_m}{r^{\tilde{d}+2}} \tag{32}$$

Since $\tilde{V}^{0N}_{\underline{0}}$ depends only on r, Eq. (26) can be written as

$$E = \int d^p\mathbf{x}\, \mathscr{E} \tag{33}$$

where

$$\mathscr{E} = \int \partial_m \tilde{V}^{0m}_{\underline{0}}\, d^{(D-d)}\mathbf{y} = \int_{r\to\infty} d^{(D-d-1)}\, \Sigma_m\, \tilde{V}^{0m}_{\underline{0}} \tag{34}$$

where $d^{(D-d-1)}\, \Sigma^m = r^{\tilde{d}} y^m\, d\Omega^{(D-d-1)}$. Using $\lim_{r\to\infty} H(r) = 1$, we calculate

$$\mathscr{E} = \frac{4kd}{\Delta}\, \Omega_{(D-d-1)} \tag{35}$$

This mass density, i.e., energy/(unit p-volume) agrees with the ADM mass density exactly.

In this paper, we evaluated the mass density of the general *Dp*-branes in string/*M*-theories. Though for fixed transverse radius r, the p-brane in the whole D-dimensional space is flat, the reduced mass density on it is not zero.

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